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Fractional Differential Problem of Some Type of Fractional Rational Function

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Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional derivative and a new multiplication of fractional analytic functions, we find arbitrary order fractional derivative of some type of fractional rational function. In fact, our result is a generalization of ordinary calculus result.**

Keywords: **Jumarie type of R-L fractional derivative, new multiplication, fractional analytic functions, fractional rational function.**

I. INTRODUCTION

Fractional calculus includes the derivative and integral of any real order or complex order. In the past few decades, fractional calculus has gained much attention as a result of its demonstrated applications in various fields of science and engineering such as physics, biology, mechanics, electrical engineering, viscoelasticity, dynamics, control theory, modelling, economics, and so on [1-11].

However, fractional calculus is different from ordinary calculus. The definition of fractional derivative is not unique. Common definitions include Riemann Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative and Jumarie's modification of R-L fractional derivative [12-16]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on the Jumarie's modified R-L fractional calculus and a new multiplication of fractional analytic functions, we find arbitrary order α-fractional derivative of the following α-fractional rational function:

$$f_\alpha(x^\alpha)=\left[\left(s\frac{1}{\Gamma(\alpha+1)}x^\alpha+t\right)^{\otimes_\alpha 2}-r^2\right]^{\otimes_\alpha(-p)}\otimes_\alpha\left[\sum_{n=0}^{\lfloor p/2\rfloor}\binom{p}{2n}r^{2n}\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha+t\right]^{\otimes_\alpha(p-2n)}\right],$$

where $0<\alpha\le 1$, s,t,r are real numbers, and p is a positive integer. In fact, our result is a generalization of classical calculus result.

II. PRELIMINARIES

At first, we introduce the fractional derivative used in this paper and its properties.

Definition 2.1 ([17])**:** Let $0<\alpha\le 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^\alpha\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^\alpha}dt\,. \tag{1}$$

where $\Gamma(\ )$ is the gamma function. On the other hand, for any positive integer m, we define $\left({}_{x_0}D_x^\alpha\right)^m[f(x)]=\left({}_{x_0}D_x^\alpha\right)\left({}_{x_0}D_x^\alpha\right)\cdots\left({}_{x_0}D_x^\alpha\right)[f(x)]$, the m-th order α-fractional derivative of $f(x)$.

Proposition 2.2 ([18])**:** *If* α, β, x_0, C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{2}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{3}$$

Next, we introduce the definition of fractional analytic function.

Definition 2.3 ([19])**:** If x, x_0, and a_k are real numbers for all k, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

Next, a new multiplication of fractional analytic functions is introduced below.

Definition 2.4 ([20])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{4}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} . \tag{5}$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \tag{6}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} . \end{aligned} \tag{7}$$

Definition 2.5 ([21])**:** If $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}, \tag{8}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \tag{9}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha\big(g_\alpha(x^\alpha)\big) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\big(g_\alpha(x^\alpha)\big)^{\otimes_\alpha n}, \tag{10}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha\big(f_\alpha(x^\alpha)\big) = \sum_{n=0}^{\infty} \frac{b_n}{n!}\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha n}. \tag{11}$$

Definition 2.6 ([22])**:** Let $0<\alpha\le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha n}=f_\alpha(x^\alpha)\otimes_\alpha\cdots\otimes_\alpha f_\alpha(x^\alpha)$ is called the nth power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)=1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha -1}$.

Notation 2.7: If r is a real number and n is a positive integer. Define $[r]_n=r(r+1)\cdots(r+n-1)$ and $[r]_0=1$.

Notation 2.8: If s is a real number, the largest integer less than or equal to s is denoted by $\lfloor s\rfloor$.

Theorem 2.9 (fractional binomial theorem)**:** *If $0<\alpha\le 1$, n is a positive integer and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions. Then*

$$[f_\alpha(x^\alpha)+g_\alpha(x^\alpha)]^{\otimes_\alpha n}=\sum_{k=0}^{n}\binom{n}{k}\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha k}\otimes_\alpha\left(g_\alpha(x^\alpha)\right)^{\otimes_\alpha (n-k)}\ ,\tag{12}$$

where $\binom{n}{k}=\frac{n!}{k!(n-k)!}$.

III. MAIN RESULTS

In this section, we find the fractional derivatives of some type of fractional rational function. At first, we need a lemma.

Lemma 3.1: *If $0<\alpha\le 1$, s,t,r are real numbers, and p is a positive integer, then*

$$\frac{1}{2}\left\{\left[\left(s\frac{1}{\Gamma(\alpha+1)}x^\alpha+t\right)+r\right]^{\otimes_\alpha(-p)}+\left[\left(s\frac{1}{\Gamma(\alpha+1)}x^\alpha+t\right)-r\right]^{\otimes_\alpha(-p)}\right\}$$

$$=\left[\left(s\frac{1}{\Gamma(\alpha+1)}x^\alpha+t\right)^{\otimes_\alpha 2}-r^2\right]^{\otimes_\alpha(-p)}\otimes_\alpha\left[\sum_{n=0}^{\lfloor p/2\rfloor}\binom{p}{2n}r^{2n}\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha+t\right]^{\otimes_\alpha(p-2n)}\right].\tag{13}$$

Proof By fractional binomial theorem,

$$\frac{1}{2}\left\{\left[\left(s\frac{1}{\Gamma(\alpha+1)}x^\alpha+t\right)+r\right]^{\otimes_\alpha(-p)}+\left[\left(s\frac{1}{\Gamma(\alpha+1)}x^\alpha+t\right)-r\right]^{\otimes_\alpha(-p)}\right\}$$

$$=\frac{1}{2}\left\{\left[\left(s\frac{1}{\Gamma(\alpha+1)}x^\alpha+t\right)^{\otimes_\alpha 2}-r^2\right]^{\otimes_\alpha(-p)}\otimes_\alpha\left[\left[\left(s\frac{1}{\Gamma(\alpha+1)}x^\alpha+t\right)+r\right]^{\otimes_\alpha p}+\left[\left(s\frac{1}{\Gamma(\alpha+1)}x^\alpha+t\right)-r\right]^{\otimes_\alpha p}\right]\right\}$$

$$=\frac{1}{2}\left\{\left[\left(s\frac{1}{\Gamma(\alpha+1)}x^\alpha+t\right)^{\otimes_\alpha 2}-r^2\right]^{\otimes_\alpha(-p)}\otimes_\alpha\left[\sum_{n=0}^{p}\binom{p}{n}r^n\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha+t\right]^{\otimes_\alpha(p-n)}\right]+\sum_{n=0}^{p}\binom{p}{n}(-r)^n\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha+\right.\right.$$

t⊗α(p−n)

$$=\frac{1}{2}\left\{\left[\left(s\frac{1}{\Gamma(\alpha+1)}x^\alpha+t\right)^{\otimes_\alpha 2}-r^2\right]^{\otimes_\alpha(-p)}\otimes_\alpha\left[\sum_{n=0}^{p}\binom{p}{n}[1+(-1)^n]r^n\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha+t\right]^{\otimes_\alpha(p-n)}\right]\right\}$$

$$=\left[\left(s\frac{1}{\Gamma(\alpha+1)}x^\alpha+t\right)^{\otimes_\alpha 2}-r^2\right]^{\otimes_\alpha(-p)}\otimes_\alpha\left[\sum_{n=0}^{\lfloor p/2\rfloor}\binom{p}{2n}r^{2n}\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha+t\right]^{\otimes_\alpha(p-2n)}\right].$$ q.e.d.

Theorem 3.2: *If $0<\alpha\le 1$, s,t,r are real numbers, and m,p are positive integers, then the m-th order α-fractional derivative of the α-fractional rational function*

$$f_\alpha(x^\alpha)=\left[\left(s\frac{1}{\Gamma(\alpha+1)}x^\alpha+t\right)^{\otimes_\alpha 2}-r^2\right]^{\otimes_\alpha(-p)}\otimes_\alpha\left[\sum_{n=0}^{\lfloor p/2\rfloor}\binom{p}{2n}r^{2n}\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha+t\right]^{\otimes_\alpha(p-2n)}\right]\tag{14}$$

is

$\left({}_0D_x^\alpha\right)^m[f_\alpha(x^\alpha)]$

$$= (-1)^m s^m [p]_m \left\{ \left[\left(s \frac{1}{\Gamma(\alpha+1)} x^\alpha + t \right)^{\otimes_\alpha 2} - r^2 \right]^{\otimes_\alpha (-p-m)} \otimes_\alpha \left[\sum_{k=0}^{\lfloor (p+m)/2 \rfloor} \binom{p+m}{2k} r^{2p} \left(s \frac{1}{\Gamma(\alpha+1)} x^\alpha + t \right)^{\otimes_\alpha (p+m-2k)} \right] \right\}. \tag{15}$$

Proof By Lemma 3.1

$$\left({}_0D_x^\alpha \right)^m [f_\alpha(x^\alpha)]$$

$$= \left({}_0D_x^\alpha \right)^m \left[\left[\left(s \frac{1}{\Gamma(\alpha+1)} x^\alpha + t \right)^{\otimes_\alpha 2} - r^2 \right]^{\otimes_\alpha (-p)} \otimes_\alpha \left[\sum_{n=0}^{\lfloor p/2 \rfloor} \binom{p}{2n} r^{2n} \left[s \frac{1}{\Gamma(\alpha+1)} x^\alpha + t \right]^{\otimes_\alpha (p-2n)} \right] \right]$$

$$= \left({}_0D_x^\alpha \right)^m \left[\frac{1}{2} \left\{ \left[\left(s \frac{1}{\Gamma(\alpha+1)} x^\alpha + t \right)^{\otimes_\alpha 2} - r^2 \right]^{\otimes_\alpha (-p)} \otimes_\alpha \left[\sum_{n=0}^{p} \binom{p}{n} [1 + (-1)^n] r^n \left[s \frac{1}{\Gamma(\alpha+1)} x^\alpha + t \right]^{\otimes_\alpha (p-n)} \right] \right\} \right]$$

$$= \frac{1}{2} \left({}_0D_x^\alpha \right)^m \left[\left[\left(s \frac{1}{\Gamma(\alpha+1)} x^\alpha + t \right) + r \right]^{\otimes_\alpha (-p)} + \left[\left(s \frac{1}{\Gamma(\alpha+1)} x^\alpha + t \right) - r \right]^{\otimes_\alpha (-p)} \right]$$

$$= \frac{1}{2} s^m (-1)^m [p]_m \left\{ \left[\left(s \frac{1}{\Gamma(\alpha+1)} x^\alpha + t \right) + r \right]^{\otimes_\alpha (-p-m)} + \left[\left(s \frac{1}{\Gamma(\alpha+1)} x^\alpha + t \right) - r \right]^{\otimes_\alpha (-p-m)} \right\}$$

$$= \frac{1}{2} s^m (-1)^m [p]_m \left\{ \left[\left(s \frac{1}{\Gamma(\alpha+1)} x^\alpha + t \right)^{\otimes_\alpha 2} - r^2 \right]^{\otimes_\alpha (-p-m)} \otimes_\alpha \left[\left[\left(s \frac{1}{\Gamma(\alpha+1)} x^\alpha + t \right) + r \right]^{\otimes_\alpha (p+m)} + \left[\left(s \frac{1}{\Gamma(\alpha+1)} x^\alpha + t \right) - \right. \right. \right.$$

$r \otimes \alpha (p+m)$

$$= (-1)^m s^m [p]_m \left\{ \left[\left(s \frac{1}{\Gamma(\alpha+1)} x^\alpha + t \right)^{\otimes_\alpha 2} - r^2 \right]^{\otimes_\alpha (-p-m)} \otimes_\alpha \left[\sum_{k=0}^{\lfloor (p+m)/2 \rfloor} \binom{p+m}{2k} r^{2p} \left(s \frac{1}{\Gamma(\alpha+1)} x^\alpha + t \right)^{\otimes_\alpha (p+m-2k)} \right] \right\}.$$

q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional calculus and a new multiplication of fractional analytic functions, we obtain arbitrary order fractional derivative of some type of fractional rational function. In fact, our result is a generalization of classical calculus result. In the future, we will continue to use Jumarie type of R-L fractional calculus and a new multiplication of fractional analytic functions to study the problems in fractional differential equations and applied mathematics.

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